First order sentences about random graphs: small number of alternations

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The random graph $G(n,p)$ *obeys zero-one law w.r.t. a first order sentence* ϕ, if either a.a.s. $G(n,p) \models \phi$, or a.a.s. $G(n,p) \models \neg(\phi)$. For each first order sentence ϕ consider the set of all $\alpha > 0$ such that $G(n, n^{-\alpha})$ does not obey zero-one law w.r.t. ϕ. This set is called *the spectrum of* ϕ. Our intuition is that the more complex a sentence, the more complex its spectrum. We consider two ways of measuring the complexity of a first order formula ϕ: $\mathrm{q}(\phi)$ — its quantifier depth, and $\mathrm{ch}(\phi)$ — the maximal number of quantifier alternations over all sequences of nested quantifiers in ϕ.

In 1990 [1], J. Spencer proved that there exists a first order sentence with an infinite spectrum and the quantifier depth 14. Let $q_{\min}$ be the minimal quantifier depth of a first order sentence with an infinite spectrum. The best known upper and lower bounds for $q_{\min}$ is $3 \leq q_{\min} \leq 5$ (see [2], [3]). Let $S(k)$ be the union of all $S(\phi)$ for all ϕ with the quantifier depth k. We examined the set $S(4)$ and proved that it has no limit points except possibly the points $1/2$ and $3/5$. We also proved that the spectrum of first order sentences, whose sequences of nested quantifiers are all of form $\forall\exists\forall\exists$ or $\exists\forall\exists\forall$, is finite.

Let $\mathrm{ch}_{\min}$ be the minimal number of quantifier alternations of a first order sentence with an infinite spectrum. The formula with an infinite spectra from [3] has $\mathrm{ch} = 4$, so $\mathrm{ch}_{\min} \leq 4$. We proved that every first order formula ϕ with $\mathrm{ch}(\phi) \leq 3$ has a finite spectra. Thus $\mathrm{ch}_{\min} = 4$.

References

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